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THE PROBLEM OF CONSTRAINED IMPACT†

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The general laws governing collisions between two rigid bodies when their displacements are subject to certain restrictions are discussed, and the legitimacy of using various mathematical models to describe such collisions is considered. Two types of constraint are discussed. The first—bilateral constraints—are conditional on one or two points of the body being fixed. It is shown that in the presence of dry friction the impact may be of the cut-off type, that is, the contact stresses do not disappear. Conditions are obtained for cut-off impact in terms of the geometry of the fixed points. Another peculiarity of the collisions of bodies with fixed points is the change in the physical meaning of the coefficient of restitution: it depends on the configuration of the system. The second type is represented by problems of impact when there is a unilateral constraint—one of the bodies is supported on a massive base; it is shown that dry friction at the point of support may lead to situations in which a solution is either non-existent or is non-unique, and which resemble the well-known Painlevé paradoxes. The following conclusion is reached: for an adequate description of the phenomenon of constrained impact, allowance must be made for the compliance of the colliding bodies not only directly in the impact pair, but also at points of contact with other bodies. In the general case, the use of wave theory to describe constrained impact creates immense mathematical difficulties and one must first work with simplified deformation models, which lead to systems of ordinary differential equations. Examples are considered, namely the impact of a physical pendulum on a wall and the Coriolis problem of colliding billiard balls.

The collision of rigid bodies is an aggregate of different physical processes, allowance for all of which is hardly possible. In dynamics, the impact problem reduces to determining the impulses of the impact forces. Historically speaking, three approaches to the solution of this problem have evolved: classical stereomechanics, whose foundations were laid by Huyghens and Newton, the wave theory of impact, which emerged from the work of St Venant at the end of the last century, and the method of deformable elements, already employed by d'Alembert [1]. The choice of one method or another to solve a given problem depends, on the one hand, on how realistic and accurate the results must be, and, on the other, on the available computing resources.

Analogous approaches are used to solve the more complex and important practical problem of impact in a *system of rigid bodies*. Worthy of mention among known results of a general theoretic nature is Appell's extension of Lagrange's equations to the case of a part of the variables in frictionless impact [2]. This approach, however, is not sufficient for a complete solution, and various additional assumptions are necessary even in the simplest cases. The stereomechanical approach [3] to impact problems in a system of connected bodies often assumes that the constraints imposed on the system are absolutely stiff relative to the stiffness of direct contact in an impacting pair. This assumption, however, may produce not only serious errors but also, in some cases, non-unique solutions [4]. The logical flaw in that hypothesis is the differential treatment of different points of the same body: at one point, corresponding to impact contact, local deformations and energy loss are allowed, but at another, where the body is attached to other bodies, they are ignored.

Compared with such an axiomatic approach, wave theory and discrete models do make allowance, to some degree or another, for deformations. Among the problems solved by these methods is the Boussinesq problem of a rigid body colliding with a rod clamped at the opposite end [5].

1. ESTIMATE OF THE IMPULSE IN THE COLLISION OF FREE RIGID BODIES

Let $\mathbf{Q}$ and $\mathbf{M_Q}$ denote the principal vector and principal moment of the active forces applied to a body, and $\mathbf{F}$ and $\mathbf{M_F}$ are the principal vector and principal moment of the reactions which occur on contact with other rigid bodies. The theorems on the motion of the mass centre and the variation of the angular momentum are expressed by the formulae

$$m\dot{\mathbf{V}} = \mathbf{Q}+\mathbf{F}, \quad \left(\mathbf{J\Omega}\right)^{\cdot} = \mathbf{M}_\mathbf{Q} + \mathbf{M}_\mathbf{F} \tag{1.1}$$

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where m is the mass, $\mathbf{J}$ is the central inertia tensor, $\mathbf{V}$ is the velocity of the centre of mass G and $\mathbf{\Omega}$ is the angular velocity of the body.

We will describe impact by the difference equations obtained from (1.1) by integration; it is assumed that the changes in the coordinates and the integrals of the "ordinary" forces are negligibly small [2]

$$m\Delta\mathbf{V} = \mathbf{I}, \quad \mathbf{J}\Delta\mathbf{\Omega} = \mathbf{M}_I \tag{1.2}$$

$$\mathbf{I} = \int_{t_0}^{t} \mathbf{F}(s)ds, \quad \mathbf{M}_I = \int_{t_0}^{t} \mathbf{M}_F(s)ds$$

The most familiar problem involves the collision of two *free* rigid bodies with one contact point C. Letting $\mathbf{F}$ denote the impact forces exerted on the first body by the second, we can write Eqs (1.2) as

$$m_1\Delta\mathbf{V}_1 = \mathbf{I}, \quad m_2\Delta\mathbf{V}_2 = -\mathbf{I}, \quad \mathbf{J}_1\Delta\mathbf{\Omega}_1 = \mathbf{G}_1\mathbf{C}\times\mathbf{I}, \quad \mathbf{J}_2\Delta\mathbf{\Omega}_2 = -\mathbf{G}_2\mathbf{C}\times\mathbf{I} \tag{1.3}$$

The number of variables may be reduced by two by using the formula for computing the tangential component $\mathbf{F}_t$ of the reaction

$$\mathbf{F}_t = \mathbf{f}(F_n, \mathbf{V}_C), \quad F_n = (\mathbf{F}, \mathbf{n}), \quad \mathbf{F}_t = \mathbf{F} - F_n\mathbf{n} \tag{1.4}$$
$$\mathbf{V}_C = (\mathbf{V}_1 + \mathbf{\Omega}_1\times\mathbf{G}_1\mathbf{C}) - (\mathbf{V}_2 + \mathbf{\Omega}_2\times\mathbf{G}_2\mathbf{C})$$

In these formulae $\mathbf{n}$ is the normal to the surfaces of the bodies at the point C, in the direction of the first body and $\mathbf{V}_C$ is the relative velocity at that point; the form of the function $\mathbf{f}$ depends on the nature of the friction. In particular, for absolutely smooth surfaces $\mathbf{f}\equiv\mathbf{0}$, and dry Coulomb friction is described by the following relation [6]

$$\mathbf{F}_t = -\mu\frac{\mathbf{V}_t}{|\mathbf{V}_t|}F_n, \quad \text{if} \quad \mathbf{V}_t \neq 0$$
$$|\mathbf{F}_t| \leqslant \mu F_n, \quad \text{if} \quad \mathbf{V}_t = 0 \tag{1.5}$$

where $\mathbf{V}_t$ and V_n are the tangential and normal components of the relative velocity, and μ is the coefficient of sliding friction (to simplify matters, we will assume that it is equal to the friction coefficient at rest).

Let us calculate the increment of the relative velocity using (1.2) and (1.3)

$$\Delta\mathbf{V}_C = \Delta\mathbf{V}_1 - \Delta\mathbf{V}_2 + \Delta\mathbf{\Omega}_1\times\mathbf{G}_1\mathbf{C} - \Delta\mathbf{\Omega}_2\times\mathbf{G}_2\mathbf{C} =$$
$$= (m_1^{-1} + m_2^{-1})\mathbf{I} + \mathbf{J}_1^{-1}(\mathbf{G}_1\mathbf{C}\times\mathbf{I})\times\mathbf{G}_1\mathbf{C} + \mathbf{J}_2^{-1}(\mathbf{G}_2\mathbf{C}\times\mathbf{I})\times\mathbf{G}_2\mathbf{C} \tag{1.6}$$

Expressing Eq. (1.6) in matrix form, we obtain

$$\Delta\mathbf{V}_C = \mathbf{BI}, \quad \mathbf{B} = (m_1^{-1} + m_2^{-1})\mathbf{E}_3 + \|b_{ij}\| \tag{1.7}$$
$$b_{ij} = \sum_{k=1}^{2}\left(\mathbf{J}_k^{-1}(\mathbf{G}_k\mathbf{C}\times\mathbf{e}_i), \mathbf{G}_k\mathbf{C}\times\mathbf{e}_j\right) \quad (i,j = 1,2,3)$$

where $\mathbf{e}_i$ are the basis vectors and $\mathbf{E}_3$ is the identity matrix of order 3.

Differentiating Eq. (1.7) with respect to the variable $\chi = I_n$ and using (1.4), we obtain a system of ordinary third-order differential equations for the impact, with initial condition $\mathbf{V}_C(t_0) = \mathbf{V}^-$. The properties of integral curves for the case of dry friction (1.5) have been discussed in [6–8] and elsewhere. To determine the time at which the impact ends, one must prescribe a boundary condition, whose form depends on the model used for the contact stresses.

In the classical theory, where no allowance is made for the deformation of the colliding bodies, the boundary condition involves the coefficient of restitution, which equals the ratio of the normal components of the impact impulse in the two phases of the impact [6]. The equations constructed above have an analytical solution in certain special cases (plane-parallel motion and the collision of two balls), but in the general case the numerical integration is necessary. Despite the fact that the model does not always agree with experiment [9–11], it is widely used in investigations of systems with impact.

Discrete models of impact, including the method of deformable elements, are more realistic. The idea of the last-named method is to place an imaginary object of zero mass and diameter, satisfying a given stress–strain relationship $\mathbf{F}(\mathbf{q}, \dot{\mathbf{q}})$, at the point of contact [1, 9]. The boundary condition then states that F_n is not negative. To check that it holds, one has to solve the system obtained by differentiating Eq. (1.7) with respect to time

$$\dot{\mathbf{V}}_C = \mathbf{BF} \tag{1.8}$$

A weakness of this approach is that the function $\mathbf{F}(\mathbf{q}, \dot{\mathbf{q}})$ is undefined.

The wave theory of impact, in which allowance is made for the deformations in the entire volume of the colliding bodies, is the most complicated approach. To verify the boundary condition $F_n \geqslant 0$, one must solve a system of partial differential equations. The problem can be solved analytically only in exceptional special cases: the collinear collision of rods or rectangular plates [9, 12], but in the general case even numerical methods do not yield visible results. Some estimates have shown [13] that wave phenomena play only a minor role in the formation of the impact impulse in collisions of bodies with non-degenerate measurements.

The different impact models share the property, proved below, that the impact impulse is bounded.

Proposition 1. When two free rigid bodies with smooth or rough surfaces collide, the impulse satisfies the following upper estimate

$$|\mathbf{I}| \leqslant 2|\mathbf{V}^-|/\lambda \tag{1.9}$$

where λ is the minimum eigenvalue of the matrix $\mathbf{B}$ in formula (1.7).

Proof. The matrix $\mathbf{B}$ is symmetric and positive-definite, as follows from the corresponding properties of the inertia tensors $\mathbf{J}_k$, and therefore $\lambda > 0$.

The increment of total kinetic energy of the bodies is defined by Kelvin's formula

$$\Delta T = \tfrac{1}{2}(\mathbf{V}^- + \mathbf{V}^+, \mathbf{I}) \tag{1.10}$$

where $\mathbf{V}^-$ and $\mathbf{V}^+$ are relative velocities at the point of contact at the beginning and end of the impact, respectively. Using (1.7), we obtain

$$\Delta T = (\mathbf{V}, \mathbf{I}) + \tfrac{1}{2}(\mathbf{BI}, \mathbf{I}) \tag{1.11}$$

In mechanical impact, no energy is released, and the kinetic energy does not increase

$$\Delta T \leqslant 0 \tag{1.12}$$

In the space $\mathbf{I} \in \mathbb{R}^3$ inequality (1.12) defines the interior of an ellipsoid passing through the origin, and this implies that the impact impulse is indeed bounded. The truth of estimate (1.9) on the boundary of this ellipsoid follows from the Cauchy–Bunyakovskii inequality.

Corollary. When the collision between free rigid bodies is completed, there are no contact stresses. Indeed, otherwise the impact impulse would have to increase without limit.

Remark. By analogy with Proposition 1, one can construct a lower estimate for the impact impulse. To that end, one uses the inequality $(\mathbf{V}^+, \mathbf{n}) \geqslant 0$, where $\mathbf{n}$ is the unit vector normal to the surfaces of the colliding bodies at the contact point. Using formula (1.7), we deduce that

$$(\mathbf{V}^-, \mathbf{n}) + (\mathbf{BI}, \mathbf{n}) \geqslant 0$$

Consequently

$$|\mathbf{I}| \geqslant |(\mathbf{V}^-, \mathbf{n})|/\Lambda \tag{1.13}$$

where Λ is the maximum eigenvalue of B.

2. COLLISION OF BODIES WITH FIXED POINT

We will now investigate collisions of bodies with fixed points. In such cases, together with the impulse directly in the impacting pair, the imposed constraints exert impact reactions on the bodies. These reactions may be expressed in terms of **I** by using the conditions that the points at which the bodies are fixed do not move (the bodies are assumed to be absolutely rigid). Finally one obtains a relation similar to (1.7)

$$\Delta \mathbf{V}_C = \mathbf{B}^* \mathbf{I} \tag{2.1}$$

The matrix $\mathbf{B}^*$ is non-negative. Indeed, if we put $\mathbf{V}^- = \mathbf{0}$ in (1.10), the left-hand side will be the kinetic energy acquired by the bodies at rest owing to the impulse. Consequently

$$\left(\mathbf{B}^*\mathbf{I}, \mathbf{I}\right) \geqslant 0 \tag{2.2}$$

Unlike the case of collisions of free bodies, the matrix $\mathbf{B}^*$ may be singular. This is because not every impulse will change the velocity of a body with fixed points. In that case the right-hand side of (1.9) goes to infinity, and Proposition 1 is not applicable to the description of constrained impact. Situations exist in which such an impact has a cut-off nature: the relative velocity at the contact point disappears, but the contact stresses persist, that is, the colliding bodies are wedged together. From a formal point of view, the duration of a cut-off impact and the impact impulse are infinitely large.

Example. Consider a rigid body rotating about a fixed axis and colliding with a massive wall. The theorem on the variation of the angular momentum in impulsive motion [2] is expressed through the formula

$$m\rho^2 \Delta \mathbf{\Omega} = (\mathbf{OC} \times \mathbf{I}, \mathbf{e})\mathbf{e} \tag{2.3}$$

where ρ is the radius of inertia about the axis of rotation, $\mathbf{e}$ is a unit vector on the axis of rotation and O is a certain fixed point. Hence, using formula (1.4), we obtain

$$\Delta \mathbf{V}_C = \Delta \mathbf{\Omega} \times \mathbf{OC} = m^{-1}\rho^{-2}\left(\mathbf{e}^*, \mathbf{I}\right)\mathbf{e}^*, \quad \mathbf{e}^* = \mathbf{e} \times \mathbf{OC} \tag{2.4}$$

Choose unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ on the surface of the wall and $\mathbf{e}_3$ orthogonal to the surface (in the direction of the first body), and let α_1, α_2 and α_3 denote the coordinates of the vector $\mathbf{e}^*$. Transform formula (2.4) to the form (2.1) with

$$\mathbf{B}^* = m^{-1}\rho^{-2} \begin{Vmatrix} \alpha_1^2 & \alpha_1\alpha_2 & \alpha_1\alpha_3 \\ \alpha_1\alpha_2 & \alpha_2^2 & \alpha_2\alpha_3 \\ \alpha_1\alpha_3 & \alpha_2\alpha_3 & \alpha_3^2 \end{Vmatrix} \tag{2.5}$$

This matrix has two eigenvalues equal to zero, while the third is positive. Cut-off impact occurs under conditions of friction (1.5), provided the friction coefficient is large enough to prevent the pendulum sliding on the obstacle in the direction opposite to the initial direction: $\alpha_3\mu \geqslant (\alpha_1^2 + \alpha_2^2)^{1/2}$.

In this example, cut-off impact turns out to be possible because there are directions passing through the contact point and the fixed axis: an impulsive reaction acting in such a direction does not make the body move. Analogous considerations yield an easily checked necessary condition for the existence of cut-off impact in the general case of collisions of two bodies with fixed points.

Let γ_j ($j = 1, 2$) denote the set of fixed points for each body. This set is a single point, a straight line or all of three-dimensional space, depending on the type of attachment.

Proposition 2. The matrix $\mathbf{B}^*$ in Eq. (2.1) is singular if and only if a straight line exists passing through the point C and intersecting both sets γ_1 and γ_2.

Indeed, a vector along such a straight line is obviously an eigenvector of $\mathbf{B}^*$, corresponding to zero eigenvalue.

Remark. System (2.1) was obtained on the assumption that the velocities of the fixed points vanish. This is the case for absolutely rigid bodies but not in deformable bodies. Nevertheless, in the latter case too, singularity of the matrix B^* is a necessary condition for cut-off impact, since then, for sufficiently large values of the friction coefficient, the system has an equilibrium position with non-zero contact stresses. To determine whether this position is attained under given impact conditions, one must set up the equations, adopt some model of the deformations, and integrate the equations.

Another important property of constrained impact is the changed meaning of the coefficient of restitution. In the classical sense, this coefficient, which is the ratio of the normal components of the impact impulse in the two phases of the impact, depends only on the materials of which the colliding bodies are made. To explain the dependence, observed in practice, of these coefficients on the shape of the bodies and the impact velocity, allowance must be made for contact deformations (see, e.g., [9, 13]). This is all the more necessary in constrained impact problems (in particular, in cut-off impact the coefficient of restitution is zero).

A model which is sufficiently simple for analysis, through which the role of the elasticity of the attachment can be ascertained, may be constructed using deformable elements.

The following example is instructive in that context.

Example. Consider the two-dimensional problem of a pendulum impacting on a smooth wall. We will assume that the pendulum and the all are absolutely rigid and that the suspension point O^* is fixed but not necessarily identical with a point O fixed in the body. Let us mentally place a deformable element Ξ_C at the point of impact contact and an element Ξ_O between the points O and O^* (Fig. 1).

To set up the equations of impulsive motion, we will take into consideration only principal terms, disregarding, in particular, the change in the orientation of the body during impact. Formulae (1.1) become

$$m\dot{\mathbf{V}} = \mathbf{F}_C + \mathbf{F}_O, \quad m\rho^2\dot{\mathbf{\Omega}} = \mathbf{GC}\times\mathbf{F}_C + \mathbf{GO}\times\mathbf{F}_O \tag{2.6}$$

where ρ is the central radius of inertia and $\mathbf{F}_C$ and $\mathbf{F}_O$ are the forces in the deformable elements

$$\mathbf{F}_C = \mathbf{F}_C\left(\varepsilon_C, \dot{\varepsilon}_C\right), \quad \mathbf{F}_O = \mathbf{F}_O\left(\varepsilon_O, \dot{\varepsilon}_O\right), \quad \varepsilon_O = -\Delta\mathbf{r}_O, \quad \varepsilon_C = -\Delta\mathbf{r}_C \tag{2.7}$$

As the contact is unilateral, we have $\mathbf{F}_O = \mathbf{0}$ if $(\varepsilon_O, \mathbf{n}) \leqslant 0$. In formulae (2.7), $\Delta\mathbf{r}$ denotes the displacement of the specified point and ε the deformation.

Together with Euler's formula, Eqs (2.6) and (2.6) yield an eighth-order system of ordinary differential equations with initial and boundary conditions

$$\begin{aligned}
&m\ddot{\varepsilon}_C = -(\mathbf{F}_C + \mathbf{F}_O) + \rho^{-2}\left[\mathbf{GC}(\mathbf{GC}, \mathbf{F}_C) - \mathbf{F}_C(\mathbf{GC}, \mathbf{GC}) + \mathbf{GO}(\mathbf{GC}, \mathbf{F}_O) - \mathbf{F}_O(\mathbf{GO}, \mathbf{GC})\right] \\
&m\ddot{\varepsilon}_O = -(\mathbf{F}_C + \mathbf{F}_O) + \rho^{-2}\left[\mathbf{GC}(\mathbf{GO}, \mathbf{F}_C) - \mathbf{F}_C(\mathbf{GO}, \mathbf{GC}) + \mathbf{GO}(\mathbf{GO}, \mathbf{F}_O) - \mathbf{F}_O(\mathbf{GO}, \mathbf{GO})\right] \\
&\varepsilon_O(t_0) = \varepsilon_C(t_0) = \mathbf{0}, \quad \dot{\varepsilon}_O(t_0) = \mathbf{0}, \quad \dot{\varepsilon}_C(t_0) = -\mathbf{V}_C(t_0) \\
&\left(\varepsilon_C(t_0 + \tau), \mathbf{n}\right) = 0
\end{aligned} \tag{2.8}$$

If the functions $\mathbf{F}_C$ and $\mathbf{F}_O$ are given, the solution of the impact problem reduces to integrating system (2.8), which can be done in the general case only by numerical means. One then obtains the velocities $\mathbf{V}_C(t_0 + \tau)$ and

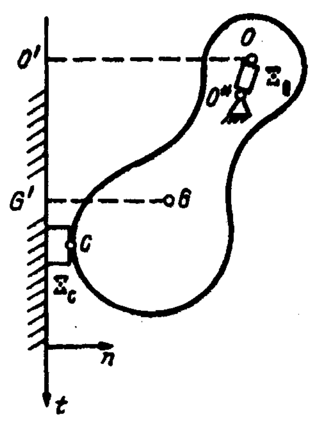

Fig. 1.

$\mathbf{V}_O(t_0 + \tau)$ of two points of the body at the end of the impact. Hence one can determine the velocity distribution at all points of the body, by solving the algebraic system

$$\begin{aligned} &\mathbf{V}(t_0+\tau)+\boldsymbol{\Omega}(t_0+\tau)\times\mathbf{GC}=\mathbf{V}_C(t_0+\tau)\\ &\mathbf{V}(t_0+\tau)+\boldsymbol{\Omega}(t_0+\tau)\times\mathbf{GO}=\mathbf{V}_O(t_0+\tau)\end{aligned} \tag{2.9}$$

The non-zero velocity at the suspension point indicates that the vibrations in the vicinity of that point persist. These vibrations are rapidly damped, during which time the kinetic energy of the body decreases. The process may be viewed as rapid braking of the point O of the body. Thus, after impact, the body may again be treated as a pendulum.

If

$$\mathbf{F}_C\perp\mathbf{GO},\ \rho^2+(\mathbf{GO},\ \mathbf{GC})=0 \tag{2.10}$$

neither deformation nor impact reaction occurs at the suspension point [14].

Since the support is smooth, the first condition in (2.10) means that the suspension point is at the same distance from the wall as the centre of inertia. The second condition imposes a restriction on the position of the contact points.

If at least one of conditions (2.10) does not hold, the solution of the problem of an impacting pendulum requires numerical integration of system (2.8). This yields the quantities $\varepsilon'_{Ot}(t_0 + \tau)$, $\varepsilon_{Cn}(t_0 + \tau)$ and $\varepsilon'_{Cn}(t_0 + \tau) = 0$. One can then calculate the velocity distribution at all points of the body by solving system (2.9), which in this case is

$$\begin{aligned} &V_{Gn}+(\mathbf{GC},\ \mathbf{t})\Omega=V_{Cn}=-\varepsilon'_{Cn}(t_0+\tau)\\ &V_{Gn}+(\mathbf{GO},\ \mathbf{t})\Omega=V_{On}=-\varepsilon'_{On}(t_0+\tau)\\ &V_{Gt}-(\mathbf{GO},\ \mathbf{n})\Omega=V_{Ot}=-\varepsilon'_{Ot}(t_0+\tau)\end{aligned} \tag{2.11}$$

where $\mathbf{n}$ and $\mathbf{t}$ are vectors normal and parallel to the surface of the obstruction.

The solution has the form

$$\begin{aligned} &\Omega(t_0+\tau)=\left[\varepsilon'_{On}(t_0+\tau)-\varepsilon'_{Cn}(t_0+\tau)\right]/[(\mathbf{GC},\ \mathbf{t})-(\mathbf{GO},\ \mathbf{t})]\\ &V_{Gt}=(\mathbf{GO},\ \mathbf{n})\Omega-\varepsilon'_{Ot}(t_0+\tau),\quad V_{Gn}=-(\mathbf{GO},\ \mathbf{t})\Omega-\varepsilon'_{On}(t_0+\tau)\end{aligned} \tag{2.12}$$

The dissipation of energy as the point O comes to a stop is calculated by formulae (1.10)

$$\Delta T=-\tfrac{1}{2}\left(V_{Ot}^2+V_{On}^2\right)+\tfrac{1}{2}[(\mathbf{GO},\ \mathbf{n})V_{Ot}-(\mathbf{GO},\ \mathbf{t})V_{On}]^2/\rho_O^2 \tag{2.13}$$

where ρ_0 is the radius of inertia of the pendulum about the suspension point.

Let us consider a special case of this problem when $m = 1$, $|GO| = 1$, $|G'C| = 1$ (in which case $(\mathbf{GO}, \mathbf{GC}) = -1$), $|GG'| = |OO'|$, where G' and O' are the projections of the points G and O, respectively, onto the obstruction (see Fig. 1). The characteristics of the deformable elements are assumed to be linear, as we set

$$\mathbf{F}_C=c_1(\boldsymbol{\varepsilon}_C,\ \mathbf{n})\mathbf{n},\quad \mathbf{F}_O=c_2\boldsymbol{\varepsilon}_O \tag{2.14}$$

Since the wall is absolutely smooth, Eqs (2.8) do not involve the variable ε_{Ct}; in addition, since the vectors $\mathbf{OG}$ and $\mathbf{F}_C$ are orthogonal, it follows that $\varepsilon_{Ot} \equiv 0$. Therefore, the order of system (2.8) can be reduced to four and it may be written as

$$\begin{aligned} &\varepsilon''_{On}=-\varepsilon_{On}\left(1+\rho^{-2}\right)-\sigma\varepsilon_{Cn}\left(1-\rho^{-2}\right)\\ &\varepsilon''_{Cn}=-\varepsilon_{On}\left(1-\rho^{-2}\right)-\sigma\varepsilon_{Cn}\left(1+\rho^{-2}\right),\quad \sigma=c_1/c_2\\ &\varepsilon_{On}(t_0)=\varepsilon_{Cn}(t_0)=0,\ \ \varepsilon'_{On}(t_0)=0,\ \ \varepsilon'_{Cn}(t_0)=1,\ \ \varepsilon_{Cn}(t_0+\tau)=0\end{aligned} \tag{2.15}$$

Figure 2 shows graphs of the relative energy loss $\Delta T/T_0$ as a function of the ratio of the stiffnesses σ for two values of the parameter ρ. Computations show that for $\rho = 0.5$ maximum loss occurs at $\sigma = 1$ (about 36% of the total kinetic energy of the pendulum); when $\rho = 2$, this again occurs at $\sigma = 1$ (about 53%). At ρ values in the intervals (0, 0.3), (0.7, 1.4), (3, $+\infty$) and $\sigma \in (0, 5)$, the energy loss is at most 15%.

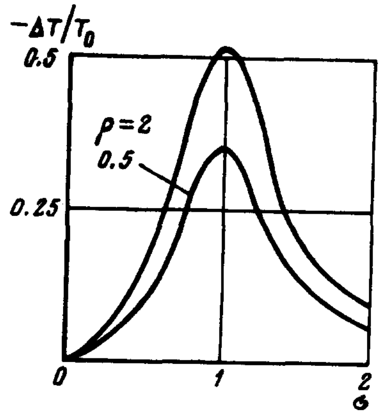

Fig. 2.

3. COLLISION IN THE CASE OF AN IDEAL UNILATERAL CONSTRAINTS

Another type of constrained impact problem arises when the system involves unilateral constraints. We will limit the discussion to the simplest case, namely, the collision of two rigid bodies, one of which is free and the other in contact at a point A with a fixed surface (at the starting time of the impact the body may be at rest, rolling along the surface or sliding along it).

Denote the reaction of the support by $\mathbf{F}^*$. By analogy with (1.3), the equations of impulsive motion are as follows:

$$m_1\Delta\mathbf{V}_1 = \mathbf{I}, \quad \mathbf{J}_1\Delta\mathbf{\Omega}_1 = \mathbf{G}_1\mathbf{C}\times\mathbf{I}$$
$$m_2\Delta\mathbf{V}_2 = -\mathbf{I}+\mathbf{I}^*, \quad \mathbf{J}_2\Delta\mathbf{\Omega}_2 = -\mathbf{G}_2\mathbf{C}\times\mathbf{I}+\mathbf{G}_2\mathbf{A}\times\mathbf{I}^* \tag{3.1}$$

where A is the point of contact with the support. The fact that the constraint is unilateral is represented by the inequality

$$(\mathbf{F}^*, \mathbf{N}) \geqslant 0 \tag{3.2}$$

which states that the normal component of the reaction of the surface is non-negative ($\mathbf{N}$ is the normal at A). Inequality (3.2) means that the body does not adhere to the surface.

The single equation (3.2) is, of course, insufficient to determine the reaction of the plane as a function of $\mathbf{F}$; certain additional assumptions are necessary. In the classical approach, one assumes [2, 15] that the normal component of the acceleration at A must be non-negative, the existence of a non-vanishing reaction at that point indicating that the contact is maintained

$$(\dot{\mathbf{V}}_A,\mathbf{N}) \geqslant 0, \quad \mathbf{F}^*(\dot{\mathbf{V}}_A,\mathbf{N}) \equiv 0 \tag{3.3}$$

The equality in (3.3) is often referred to as the complementarity condition.

By specifying the nature of the friction at the points A and C, one obtains a model of constrained impact of absolutely rigid bodies.

We will now discuss the suitability of that model for solving problems in mechanics.

Transform system (3.1) as follows:

$$\dot{\mathbf{V}}_C = \mathbf{BF} - \tilde{\mathbf{B}}'\mathbf{F}^*, \quad \dot{\mathbf{V}}_A = \mathbf{B}_A\mathbf{F}^* - \tilde{\mathbf{B}}\mathbf{F}$$
$$\tilde{\mathbf{B}}\mathbf{F} = m_2^{-1}\mathbf{F} + \mathbf{J}_2^{-1}(\mathbf{G}_2\mathbf{C}\times\mathbf{F})\times\mathbf{G}_2\mathbf{A}, \quad \mathbf{B}_A\mathbf{F}^* = m_2^{-1}\mathbf{F}^* + \mathbf{J}_2^{-1}(\mathbf{G}_2\mathbf{A}\times\mathbf{F}^*)\times\mathbf{G}_2\mathbf{A} \tag{3.4}$$

The matrix $\mathbf{B}$ is as defined in (1.7) and the prime denotes transposition. Note that $\mathbf{B}_A$ is symmetric and positive-definite.

Evaluate the scalar product of both sides of the second equation in (3.4) by $\mathbf{N}$

$$(\dot{\mathbf{V}}_A,\mathbf{N}) = (\mathbf{B}_A\mathbf{F}^*,\mathbf{N}) - (\tilde{\mathbf{B}}\mathbf{F},\mathbf{N}) \tag{3.5}$$

Let us assume that the surface is absolutely smooth, that is, the tangential component of the reaction at A vanishes (when that happens there may be friction at the point C of contact between the bodies), i.e. $\mathbf{F}^* = F_N^*\mathbf{N}$. Then Eq. (3.5) may be written as

$$(\dot{\mathbf{V}}_A, \mathbf{N}) = \alpha + \beta F_N^*, \quad \alpha = -(\tilde{\mathbf{B}}\mathbf{F}, \mathbf{N}), \quad \beta = (\mathbf{B}_A\mathbf{N}, \mathbf{N}) \tag{3.6}$$

The fact that the coefficient β in Eq. (3.6) is positive guarantees the uniqueness of a solution satisfying conditions (3.2) and (3.3). The solution is

$$F_N^* = \max\{0, -\alpha/\beta\} \tag{3.7}$$

By the definition of the coefficients α and β, F_N^* is a piecewise-linear function of $\mathbf{F}$; substituting (3.7) into the first formula of (3.4), we obtain

$$\dot{\mathbf{V}}_C = \begin{cases} \mathbf{BF}, & \text{if} \quad \alpha \geqslant 0 \\ \mathbf{B}^*\mathbf{F}, & \text{if} \quad \alpha < 0 \end{cases}, \quad \mathbf{B}^*\mathbf{F} = \mathbf{BF} - \beta^{-1}(\tilde{\mathbf{B}}\mathbf{F}, \mathbf{N})\tilde{\mathbf{B}}'\mathbf{N} \tag{3.8}$$

The matrix $\mathbf{B}^*$ is symmetric and positive-definite. Indeed

$$(\mathbf{B}^*\mathbf{F}, \mathbf{F}) = (\mathbf{BF}, \mathbf{F}) - \beta^{-1}(\tilde{\mathbf{B}}\mathbf{F}, \mathbf{N})^2 = m_1^{-1}F^2 + (\mathbf{J}_1^{-1}(\mathbf{G}_1\mathbf{C}\times\mathbf{F}), \mathbf{G}_1\mathbf{C}\times\mathbf{F}) +$$
$$+\beta^{-1}\left\{\left[m_2^{-1} + (\mathbf{J}_2^{-1}(\mathbf{G}_2\mathbf{A}\times\mathbf{N}), \mathbf{G}_2\mathbf{A}\times\mathbf{N})\right]\left[m_2^{-1}F^2 + (\mathbf{J}_2^{-1}(\mathbf{G}_2\mathbf{C}\times\mathbf{F}), \mathbf{G}_2\mathbf{C}\times\mathbf{F})\right] -\right.$$
$$\left.-\left[m_2^{-1}F_N + (\mathbf{J}_2^{-1}(\mathbf{G}_2\mathbf{C}\times\mathbf{F}), \mathbf{G}_2\mathbf{A}\times\mathbf{N})\right]^2\right\} \geqslant m_1^{-1}F^2$$
$$(\mathbf{B}^*\mathbf{u}, \mathbf{v}) = (\mathbf{Bu}, \mathbf{v}) - \beta^{-1}(\tilde{\mathbf{B}}\mathbf{u}, \mathbf{N})(\tilde{\mathbf{B}}'\mathbf{N}, \mathbf{v}) = (\mathbf{B}^*\mathbf{v}, \mathbf{u})$$

By analogy with Proposition 1, one can prove the following.

Proposition 3. In the impact of two rigid bodies with smooth or rough surfaces, one free and the other touching a smooth massive support, the impulse has an upper limit as in (1.9), where λ is the minimum eigenvalue of the matrices $\mathbf{B}$ and $\mathbf{B}^*$ in formula (3.8).

Hence it follows that on completion of constrained impact of the type considered, the contact stresses at the point C vanish.

To solve the impact problem one can apply the scheme considered in the previous section: given the friction law at the point C, one transforms system (3.8) to a new independent variable χ and integrates the system for the given initial data. When that is done, the boundary condition is determined using the coefficient of restitution (recall that system (3.8) was obtained in the context of the classical impact theory).

Example. The problem of the collision of two balls on a billiard table was solved in [16], allowing for friction among the balls but disregarding friction between each of the balls and the cloth. The complementarity condition (3.3) was not used; instead, it was assumed that the direction of the relative velocity $\mathbf{V}_C$ remains unchanged during the impact. We will investigate that system using the method described above (incidentally justifying the approach itself, that is, showing that the impact reaction of the table to one of the balls is identically zero). Without loss of generality, we will assume that the projection of the vector $\mathbf{V}^-$ onto the vertical $\mathbf{N}$ is negative (otherwise, one need only change the numbering of the balls).

Taking the vectors $\mathbf{n}$, $\mathbf{N}\times\mathbf{n}$ and $\mathbf{N}$ as an orthonormal basis, we have

$$\mathbf{G}_1\mathbf{C} = -\mathbf{G}_2\mathbf{C} = (R, 0, 0), \quad \mathbf{G}_2\mathbf{A} = (0, 0, -R), \quad \mathbf{J}_1 = \mathbf{J}_2 = m\rho^2\mathbf{E}_3$$

where R is the radius of each ball and ρ is the radius of inertia (for a homogeneous ball, $\rho^2 = 0.4R^2$). The matrices $\mathbf{B}$ in formula (1.7) and $\mathbf{B}_A$, $\tilde{\mathbf{B}}$ in (3.4) may be expressed as follows:

$$\mathbf{B} = \frac{2}{m}\operatorname{diag}\left\{1, \frac{R^2}{\rho^2}+1, \frac{R^2}{\rho^2}+1\right\}, \quad \mathbf{B}_A = \frac{2}{m}\operatorname{diag}\left\{\frac{R^2}{\rho^2}+1, \frac{R^2}{\rho^2}+1, 1\right\} \tag{3.9}$$

$$\tilde{\mathbf{B}} = \frac{2}{m}\begin{Vmatrix} 1 & 0 & -R^2/\rho^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}$$

By assumption, we have $\mathbf{F}_N > 0$ at the starting time, and therefore the first ball experiences no impact reaction from the table. In addition, $\alpha = -F_N/m < 0$ and, by formula (3.8), the change in relative velocity at the point C is determined by the matrix

$$\mathbf{B}^* = \frac{2}{m}\operatorname{diag}\left\{2, 2\frac{R^2}{\rho^2}+2,\ 2\frac{R^2}{\rho^2}+1\right\} \tag{3.10}$$

The sliding friction is described by the first of formulae (1.5)

$$\mathbf{F} = \left(1, -\mu V_2\left(V_2^2+V_3^2\right)^{-1/2}, -\mu V_3\left(V_2^2+V_3^2\right)^{-1/2}\right)F_n$$
$$V_1 = (\mathbf{V}_C, \mathbf{n}), \quad V_2 = (\mathbf{V}_C, \mathbf{N}\times\mathbf{n}), \quad V_3 = (\mathbf{V}_C, \mathbf{N}) \tag{3.11}$$

Changing in system (3.8) to the independent variable $\chi = I_n$, we write it as follows for the case of homogeneous balls (the prime denotes differentiation with respect to χ)

$$mV_1' = 2, \quad mV_2' = -7\mu V_2\left(V_2^2+V_3^2\right)^{-1/2}, \quad mV_3' = -7\mu V_3\left(V_2^2+V_3^2\right)^{-1/2} \tag{3.12}$$

Now transform system (3.12) to polar coordinates r, φ, setting $V_2 = r\sin\varphi$, $V_3 = r\cos\varphi$ (thus, the quantity r equals the absolute value of the tangential component of the relative velocity of the balls at their point of contact and φ is the angle between that component and the vertical)

$$r'\cos\varphi - r\varphi'\sin\varphi = -6\frac{\mu}{m}\cos\varphi, \quad r'\sin\varphi + r\varphi'\cos\varphi = -7\frac{\mu}{m}\sin\varphi$$

Hence, finally

$$V_1' = \frac{2}{m}, \quad r' = -\frac{\mu}{m}(6+\sin^2\varphi), \quad r\varphi' = -\frac{1}{2m}\mu\sin 2\varphi \tag{3.13}$$

As is obvious from the third equation in (3.13), the angle φ remains constant in two cases only: $\varphi = \pm\pi$ (the relative velocity vector lies in a vertical plane containing the centres of the balls; this happens, in particular, in direct impact of a ball rolling without sliding over a fixed ball) or $\varphi = \pm\pi/2$ (the relative velocity vector is horizontal). In the general case, the angle φ increases in absolute value, so that the vector $\mathbf{V}_C$ tends to occupy the position opposite to $\mathbf{N}$. When that happens, $F_N > 0$, and then $\alpha < 0$ in (3.6), and Eqs (3.13) describe the evolution of the impact up to its completion. On the basis of these equations, one can draw certain qualitative conclusions as to the nature of the impact.

First, the quantity r' is negative. Consequently, relative sliding of the balls will always slow down and, once halted, does not recur. Second, at the instant sliding halts (if it does) one has $\varphi = \pm\pi$, that is, the vector $\mathbf{V}_C$ points vertically downward.

The solution of system (3.13) may be reduced to quadratures. To that end, divide the second equation by the third and integrate the resulting equation with separated variables. The result is

$$r = r_0\left(\frac{\sin\varphi}{\sin\varphi_0}\right)^6\left(\frac{\cos\varphi}{\cos\varphi_0}\right)^{-7} \tag{3.14}$$

Substituting (3.14) into the third of formulae (3.13), we get an equation from which φ may be found as an (explicit) function of I_n. The first equation enables one to determine I_n from the boundary condition

$$I_n = -\frac{1+e}{2}V_n^-$$

where e is the coefficient of restitution.

We have thus constructed a solution of the Coriolis problem in closed form.

The following conclusion can be drawn: the problem of the impact of two rigid bodies in the presence of an additional ideal unilateral constraint has a unique solution satisfying conditions (3.2) and (3.3).

Whether this solution is realistic can be verified through a simple experiment.

Place one of the balls at the rim and hit it with the other at the point antipodal to the rim (instead of billiard balls, one can use a pair of coins on a smooth table, one of which is touching the wall). By the second condition of (3.3), the originally immobile ball should remain immobile. In practice, the outcome is different: the ball moves away from the rim.

This result may be obtained by making allowance for elastic deformations on impact. To that end, one can place deformable elements at the points of contact of the two balls and of the ball with the wall, and set up the differential equations of impulsive motion, as was done in the previous section (see also [14]). Numerical computations for the elastic impact of two identical balls on a substantially more rigid wall lead to the following result: the rebound velocity of the uncoming ball is about 96% of its initial velocity V, while the originally motionless ball recoils from the wall at velocity $0.28V$.

Experimental verification of the solution of the Coriolis problem is difficult, since the coefficient of friction between billiard balls is extremely small (according to the data of [16], it is approximately 0.03). Numerical solution based on the method of deformable elements shows that the direction of the slip velocity may in general vary. Consequently, the hypothesis proposed in [16] is not confirmed.

4. COLLISION OF BODIES ONE OF WHICH RESTS ON A ROUGH SURFACE

Unlike the problem considered in Section 3, which has a unique solution in classical impact theory, the problem of a rigid body colliding with a body in contact with a rough support is in certain cases ill posed. This paradox is analogous to the well-known paradox of Painlevé in systems with dry friction [17]. To understand the nature of the paradox, we will try to set up an equation analogous to (3.8), assuming dry friction of type (1.5) in the support, with coefficient μ^*

$$\mathbf{F}_T = -\mu^* \frac{\mathbf{V}_T}{|\mathbf{V}_T|} F_N, \quad \text{if} \quad \mathbf{V}_T \neq \mathbf{0}$$
$$|\mathbf{F}_T| \leqslant \mu^* F_N, \quad \text{if} \quad \mathbf{V}_T = \mathbf{0} \tag{4.1}$$

The subscripts N and T indicate the normal and tangential components of the vector at the point A. We must consider four possible types of motion of that point.

1. The body becomes detached from the surface, as represented by relations $\mathbf{F}^* = \mathbf{0}$, $(\dot{\mathbf{V}}_A, \mathbf{N}) > 0$. In this situation, because of (3.6), the coefficient α is positive.

2. As the body rolls on the surface, we have conditions $\mathbf{V}_A = \mathbf{0}$, $\dot{\mathbf{V}}_A = \mathbf{0}$, $F_N^* \geqslant 0$, $|\mathbf{F}_T| \leqslant \mu^* F_N^*$. Equating the right-hand side of the second formula in (3.4) to zero, we obtain

$$\mathbf{B}_A \mathbf{F}^* = \tilde{\mathbf{B}}\mathbf{F} \tag{4.2}$$

By definition, $\mathbf{B}_A$ is symmetric and positive-definite, and so Eq. (4.2) has a unique solution $\mathbf{F}^* = \mathbf{B}_A^{-1}\tilde{\mathbf{B}}\mathbf{F}$.

3. If the body slides along the support, i.e. $\mathbf{V}_T \neq \mathbf{0}$, then

$$\mathbf{F}^* = F_N^* \mathbf{l}, \quad \mathbf{l} = \mathbf{N} - \mu^* \mathbf{e}_T, \quad \mathbf{e}_T = \mathbf{V}_T / |\mathbf{V}_T| \tag{4.3}$$

Substituting (4.3) into formula (3.5), we obtain an equation of type (3.6) in which α remains unchanged but

$$\beta = (\mathbf{B}_A \mathbf{l}, \mathbf{N}) \tag{4.4}$$

4. A transition from rolling to sliding in the direction $\mathbf{e}_T$ occurs when the following conditions hold

$$F_N^* \geqslant 0, \quad \varkappa \mathbf{e}_T = F_N^* \mathbf{B}_A (\mathbf{N} - \mu^* \mathbf{e}_T) - \tilde{\mathbf{B}}\mathbf{F} \tag{4.5}$$

where $\varkappa$ is a positive number. Here the right-hand side is the quantity $\dot{\mathbf{V}}$, evaluated by formula (3.4), taking (4.3) into account. The vector equality (4.5) contains three unknown quantities: $\varkappa$, F_N^* and the angle between $\mathbf{e}_T$ and $\mathbf{n}$; in principle, therefore, the problem of determining the direction of sliding may have a unique solution.

We will now ascertain which of the above types of motion may take place in given impact conditions. Let us first assume that the second body is sliding along the support at the starting time. If the coefficient (4.4) is positive, Eq. (3.6) has a unique solution (3.7) (depending on the sign of α, one obtains cases 1 and 3). If $\beta < 0$, Eq. (3.6) has no solutions compatible with (3.3) (if $\alpha < 0$), or two such solutions at once (if $\alpha > 0$).

If there is no slip at the point A, then, apart from the cases in which a unique solution of the first, second or third types, exist solutions of all these types may exist simultaneously. This situation is analogous to the paradoxes of the motion of a rigid body on a rough support, as considered for two dimensions in [18], where conditions for the problem to be well posed were derived.

Without going into the details of the rather lengthy analysis, we merely remark that the problem just discussed cannot always be solved within the context of classical impact theory. In order to resolve the paradoxes, one must drop conditions (3.3) and construct an impact model that takes the deformations at both points A and C into account. Analysis of such models shows that in the paradoxical situations when the solution is not unique, the body actually detaches from the support; in such cases the impact impulse at the point C may be computed by methods of stereomechanics. As to the case in which there are no solutions compatible with (3.3), one cannot avoid making allowance for the deformations.

Example. Let us consider the impact of a rolling billiard ball with a stationary ball, taking into account the roughness of the balls and the table. This problem was considered in [19] using the complementarity condition (3.3), in relation to Coriolis' hypothesis that the direction of sliding at C is invariant. As remarked in the previous section, these assumptions are compatible only in the simplest special case of direct impact.

Let us consider the situation using the scheme described previously (that is, without using Coriolis' hypothesis).

At the start of the impact the quantity $\alpha = -F_N/m$ is negative, and $\mathbf{V}_A = \mathbf{0}$. Depending on the value of μ^*, the second ball will slide along the support (the fourth case) or roll along it (the second case). Computations show that contact with the support in all cases 2–4 requires that $F_3^* = F_3$. In rolling $F_1^* = 2/7F_1 - 5/7F_3$, $F_2^* = 2/7F_2$, and moreover $(\mu^* F_3^*)^2 \geq F_1^{*2} + F_2^{*2}$.

Sliding occurs if the last inequality is inverted, and in that case the initial direction of sliding is opposite to that of the vector $(2/7F_1 - 5/7F_3, 2/7F_2, 0)$.

In sliding, by (4.3), one has $\mathbf{F}^* = F_3\mathbf{I}$.

In this example $\beta = 1/m > 0$, and therefore the problem of determining $\mathbf{F}^*$ as a function of $\mathbf{F}$ has a unique solution.

Compared to the case of a smooth support, examined in the previous section, the present problem has an essentially new property: the relative acceleration at the point of contact C of the balls depends not only on $\mathbf{F}$ but also on the relative velocity at the point A of contact of the second ball with the table. In particular, if sliding occurs at A, Eqs (3.4) hold, with $\mathbf{F}^* = F_3\mathbf{I}$. The equation for the normal component of the velocity $\mathbf{V}_C$ can be separated out, and the remaining non-linear fourth-order system may be transformed to polar coordinates, setting (as before) $V_2 = r\sin\varphi$, $V_3 = r\cos\varphi$, and also $\mathbf{e}_T = (r^*\cos\xi, r^*\sin\xi, 0)$. We finally obtain

$$\begin{aligned}
mr' &= -\mu(6+\sin^2\varphi) + \mu\mu^*\cos\varphi\left(\tfrac{5}{2}\cos\xi\cos\varphi - \sin\xi\sin\varphi\right)\\
mr\varphi' &= -\mu\sin\varphi\cos\varphi - \mu\mu^*\cos\varphi\left(\tfrac{5}{2}\cos\xi\sin\varphi + \sin\xi\cos\varphi\right)\\
mr^{*\prime} &= 7\mu^*\mu\cos\varphi - \cos\xi - \mu\left(\tfrac{5}{2}\cos\xi\cos\varphi - \sin\xi\sin\varphi\right)\\
mr^*\xi' &= \sin\xi + \mu\left(\cos\xi\sin\varphi + \tfrac{5}{2}\sin\xi\cos\varphi\right)
\end{aligned} \tag{4.6}$$

Let us see under what conditions the angle φ remains constant (that is, Coriolis' hypothesis is valid). The right-hand side of the second equation of (4.6) vanishes in three cases: $\cos\varphi = 0$ (direct impact), $\mu = 0$ or

$$\sin\varphi + \mu^*\left(\tfrac{5}{2}\cos\xi\sin\varphi + \sin\xi\cos\varphi\right) = 0 \tag{4.7}$$

For (4.7) to hold with $\mu^* \neq 0$, the angle ξ must also remain constant. This gives a system of the two equations (4.7) together with $\xi' = 0$ (the derivative is evaluated from the fourth equation of (4.6)) to determine the values of φ and ξ for such friction coefficients. The constants μ and μ^* occur linearly in this system. Hence it follows that to each pair of values φ and ξ for which the expressions in parentheses in the equations of the system do not vanish there corresponds a pair of values μ and μ^* for which the direction of sliding between the balls remains unchanged during impact. Thus, the algebraic solution obtained in [19] only holds in certain exceptional special cases.

In particular, if $\mu = \mu^*$, the desired property is obtained both in direct impact and in impact such that $\cos\varphi = -2/(7\mu)$, provided that $\mu > 2/7$ (in which case $\xi = \varphi$).

Apart from these exceptions, the direction of the impact is not conserved, and numerical integration is needed to solve system (4.6).

The above problem may be solved in the context of the classical theory, which cannot be said of the following simple example.

Examples. 1. Let us investigate the plane collision of two bodies (plates) on the assumption that there is no friction at their point of contact. We introduce a Cartesian system of coordinates OXY in such a way that the support plane is described by the equation $y = 0$, and we denote the coordinates of the vectors as follows:

$$G_1C = (a_1, b_1), \quad G_2C = (a_2, b_2), \quad G_2A = (a_3, b_3), \quad b_3 < 0$$

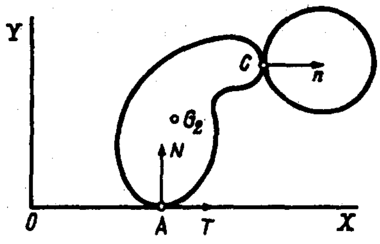

Fig. 3.

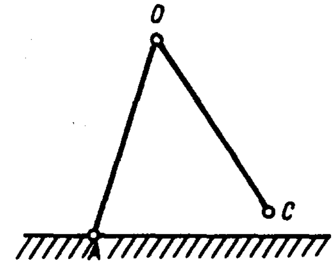

Fig. 4.

The matrices defined in (1.7) and (3.4), which define the impact scenario, have the following expressions

$$\mathbf{B}=\left(m_1^{-1}+m_2^{-1}\right)\mathbf{E}_2+m_1^{-1}\rho_1^{-2}\begin{Vmatrix} b_1^2 & -a_1b_1 \\ -a_1b_1 & a_1^2 \end{Vmatrix}+m_2^{-1}\rho_2^{-2}\begin{Vmatrix} b_2^2 & -a_2b_2 \\ -a_2b_2 & a_2^2 \end{Vmatrix}$$

$$m_2\mathbf{B}_A=\mathbf{E}_2+\rho_2^{-2}\begin{Vmatrix} b_3^2 & -a_3b_3 \\ -a_3b_3 & a_3^2 \end{Vmatrix},\quad m_2\tilde{\mathbf{B}}=\mathbf{E}_2+\rho_2^{-2}\begin{Vmatrix} b_2b_3 & -a_3b_2 \\ -a_2b_3 & a_2a_3 \end{Vmatrix}$$

We shall assume that the normal $\mathbf{n}$ is horizontal (Fig. 3).

We first assume that at the start of the impact the second body is sliding to the right along the support (in that case $\mathbf{F}^* = \mathbf{0}$); by formula (4.4),

$$m_2\rho_2^2\beta=a_3^2+\rho_2^2+\mu^*a_3b_3 \tag{4.8}$$

The quantity β in this formula may be negative, if $a_3 > 0$ and the friction coefficient is large enough. Under our assumptions $\mathbf{F} = \gamma\mathbf{n}$, $\gamma > 0$, so that in formula (3.6) we have $\alpha=\gamma m_2^{-1}\rho_2^{-2}a_3b_2$. Consequently, if $b_2 > 0$, Eq. (3.6) has two solutions, while if $b_2 < 0$ it has none.

Now suppose that at the starting time $\mathbf{V}_A = \mathbf{0}$. Computations show that if

$$a_3<0,\quad b_2<0,\quad \beta<0,\quad \left(\rho_2^2+b_2b_3\right)-b_3a_2\frac{\mu^*(\rho_2^2+b_3^2)+a_3b_3}{\rho_2^2+a_3^2+\mu^*a_3b_3}<0$$

motion of each of the three types 1, 2 and 4 is possible.

2. As regards problems of bipedal locomotion, some authors [4] have discussed a system consisting of two hinged rods ("legs") stepping alternately on a rough supporting surface (Fig. 4), and proposed looking for a solution using the complementarity condition (3.3). This has resulted in paradoxical situations, analogous to those discussed above. Although the system is somewhat different from those considered in this section, it can be investigated in a similar way and an equation of the type (3.6) obtained. The paradoxes arise when $\beta < 0$.

To resolve these paradoxes, one can use the method of deformable elements. The result is as follows: In the non-uniqueness cases, one has separation from the support ($F^*_N = 0$). Cases in which the classical model produces no solution are not exceptional when allowance is made for deformations, but they do possess a specific property: however small the ratio m_1/m_2, the deformations at the point A are not small. Qualitatively speaking, the situation is the same as in the case of "tangential impact" [17]: the impact at A is due to non-correspondence of the tangential velocities rather than normal ones. After such an impact the second body rebounds from the supporting surface ("the fly overturns the elephant").

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